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MATHEMATICS
AND
THE THEORY OF SCIENCE

BY

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KØBENHAVN
HOVEDKOMMISSIONÆR: ANDR. FRED. HØST & SØN, KGL. HOF-BOGHANDEL
BIANCO LUNOS BOGTRYKKERI
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I.

When IMMANUEL KANT, about 150 years ago, put the discouraging question to himself: Does true science, that is, perfectly reliable and exact science, really exist? he was soon able to answer: Yes, for we have mathematics. — Nobody ever entertained doubts about mathematics, not even HUME, who was a sceptic in most other things.

In our day a decision could not so easily have been arrived at, for a strange thing has happened: Doubts have arisen even about the general reliability of mathematics.

As well known, it was the Greek mathematician EUCLID who, about 300 years B. C., produced the first exact systematic statement of the fundamental mathematical elements. These "Elements" were generally accepted. From this base the great Greek mathematicians as ARCHIMEDES and APOLLONIUS proceeded, from this base again the following generations continued, and even in our day most mathematicians, or at least a great number of them, stand altogether on the Euclidian base. For such the so-called Euclidian mathematics is the entire mathematics.

But even shortly after the appearance of the Elements some slight criticism began to assert itself.

While EUCLID on the whole proceeded with great caution, he had, however, made one assertion which according to several people's opinion was somewhat too bold to be accepted as the directly evident outcome of the given definitions, the so-called Parallel-Postulate: If two straight lines be intersected by a third, in the same plane, and the two interior angles on one side be less than two right angles, then the two lines, being sufficiently produced, will meet on that side. That this was a true assertion, was hardly doubted by any efficient critic for more than two thousand years. The only objection made to it was that Euclid had not given it as a theorem, confirmed by proof. Attempts were then made to remedy this deficiency, and the remarkable thing happened that, in spite of strenuous efforts for more than two thousand years to prove the correctness of the said assertion, the melancholy result was that none of the many attempts gave universal satisfaction. As soon as somebody thought that the question had been settled, somebody else found that there was still a link missing in the demonstration, or that it was founded on another assumption, as yet unproved, and as doubtful as the first. The immediate consequence of this curious fact was that, after the lapse of the two thousand years, several mathematicians at last plucked up the courage to doubt the correctness of the parallel-postulate; they now deemed it an uncertain and arbitrary assertion which had no right to be included among the premisses of a scientific geometry. Such a geometry ought, they maintained, to be built without this assumption, and two such attempts were also made in the first half of the 19. century, one by the Russian Lobatschefsky, and another by the Hungarian Johann Bolyai. Thus the so-called Non-Euclidian geometry made its appearance, and in the eyes of the New-

Mathematicians the Euclidian system was reduced to, at best, an arbitrary and quite special mathematical system.

It is unquestionably a very peculiar event that has thus taken place in the first part of the nineteenth century, it may even be said, without exaggeration, to be the most remarkable revolution that has ever occurred in the field of human apprehension. That mathematics, the most "formidable" of all sciences, as KANT named it, should thus suddenly, after having reigned undisturbed for more than two thousand years, be reduced to something rather erroneous, which had better be exchanged for something truer, must surely fill one with astonishment, and we cannot dismiss the case with the remark that it was only a fit of romantic levity of the period which was filled with so many romantic extravagances, for the new doctrine did not disappear quickly again like the great philosophical systems of the romantic age; quite the contrary, the two mathematic camps exist to this day without the one being able to convince the other. Not even in the domains of mathematics do we thus find unanimously acknowledged truth in existence. For more than two thousand years mathematicians have quarrelled about what could be regarded as a proof and what not, and the contest is not over yet.

Still, there is a gleam of light in all this darkness. A closer examination will show us that the source of the disagreement is hardly to be sought in the field of mathematics proper. Like every other science mathematics, too, has its philosophical foundation, and it is undoubtedly in errors here that the origin of the disagreement must be looked for. But the philosophical theory of science is a comparatively new and imperfect discipline, and to detect mistakes here and put them right should hardly be any overwhelming piece of work. In the following an attempt will be made to remove the disunion by taking this course.

II.

First of all then, it has to be settled what it is we have to speak about.

Now and again a sort of approximative mathematics, an empirical discipline, has been posited, perhaps somewhat more accurate and certain than most other sciences, but not essentially different from them. Such a discipline has, however, no particular theoretical interest. It was not such a discipline Kant named formidable, and it was not such a discipline Euclid founded. The discipline which is here to be examined, is that which the late Professor Christiansen had in his mind when in his jocular way he said: "The difference between physics and mathematics is this, that in physics everything is wrong and in mathematics everything is right". In physics we say: Light is an electro-magnetic process in the ether, proceeding with a velocity of 300 000 km per sec. But properly speaking this is not correct. For the velocity may easîly be a score of meters, nay, even a score of kilometers greater or smaller, and whether light is actually an electro-magnetic process is also somewhat doubtful. Some fifty years ago it was considered to be an elastic process; Newton again had another opinion about it. Furthermore it is not at all so absolutely certain that an ether does exist, and finally it is not even absolutely certain that an objective space exists.

But how, now, about mathematics? If in a rectangular plane triangle one of the sides containing the right angle be 300 000 km, and the other 400 000 km, then the hypotenuse will be 500 000 km, and not even one millimeter more or less. — That is the language of the discipline which it is of the utmost theoretical interest to arrive at a fundamental understanding of.

For if this language be legitimate, mathematics is, in a certain sense, a perfect science, and we thus face the curious case that we imperfect human beings can create perfect science, and the interesting fundamental question: How on earth can that be possible?

That such a possibility really exists is obvious. For we all know a diminutive part of such a science, namely, the well-known multiplication table. It has no doubt existed with quite the same results for several thousand years, and we all feel perfectly convinced that if human beings still exist after the lapse of another thousand years, they will acknowledge the same quite unaltered results. Perfect science does thus exist. The next question is to find out how this is possible. Nor is this very difficult, even if this question has caused both confusion and misapprehension.

All the sciences may be divided into two kinds: the formal or ideal, and the real or empirical sciences. — The formal sciences, as logic, mathematics and rational mechanics, treat only of objects which we have created ourselves by our definitions. We therefore know them through and through. They do not possess a single property beyond those we have given them ourselves and the logical consequences of the same. Therefore they can never surprise us by any imprevisible behaviour, all will be the logical outcome of the definitions. We need never resort to experience for the verification of results. We can proceed quite rationally, quite logically, and here, therefore, we can obtain general and universally valid certainties and exactitudes.

The real sciences, as f. inst. the natural sciences, on the other hand, treat only of observed objects. Consequently they must begin with observations, measurements etc. Here, therefore, we can never be sure that we know the objects fully. We always get incomplete starting points, and so we must always verify the obtained results by fresh observations, just as we must always be prepared for imprevisible surprises. Here, therefore, we obtain only general and universally valid probabilities and approximations.

It is easy to see the propriety of this division, and that it just explains the perfection of mathematics. How was the mathematician able to speak with such perfect certainty and exactness about the triangle mentioned above? He may perhaps have commenced by drawing a rough triangle on the blackboard; but then he says: Let this represent a rightangled plane triangle with the sides 3 and 4 containing the right angle. By this remark the scene has at once been removed from the triangle on the blackboard to the mental triangle in his own and the auditors' minds. The right angle and the sides have without any trouble got just the right sizes, and he can now in a purely logical way make the auditors see the justness of his assertions. But suppose on the contrary that it was a real triangle, as f. inst. a triangular piece of brass or glass, that had to be examined. Then it would be quite impossible for us to be sure that it was a perfect plane, that the right angle measured exactly 90°, or that the sides were actually straight lines. We might apply for the assistance of mathematics itself, and carefully measure each side sixteen hundred times. We might then perhaps obtain a forty times as great accuracy in our approximate result, but only perhaps, for suppose the temperature had varied during all these measurements!

In short, with regard to observed objects it is only possible to obtain probabilities and approximations. The mathematical objects are, of course — in virtue of our silence — independent of heat and cold, light and darkness, summer and winter, youth and age etc. etc., all geometrical presuppositions, which HILBERT, who has taken upon himself to give us the perfect system of the geometrical axioms, has by the way quite neglected.

A short remark must here be offered to prevent misunderstandings. In the midst of an empirical science a fragment of a mere formal science will sometimes be introduced which will then of course follow the ordinary laws of the formal science. Thus in the science of physics or astronomy we may say: Suppose we have an empty space and in this a fixed globe of the mass m_1. At the distance r from it we suppose another globe of the mass m_2 and with such and such a velocity and direction. Let the two attract each other according to NEWTON's law, and let us seek to determine the shape and magnitude of the orbit of m_2. — As will be seen, here again we are dealing only with objects created by our own pronouncements, and we may therefore obtain a perfectly certain and exact result, that is, for this mental world. As soon as we return to real astronomy we must again content ourselves with approximations and probabilities. And exactly the same is true for the whole of the so-called applied mathematics. Its results are only valid in the same degree as the real objects and situations resemble the corresponding mathematical ones, and as to the degree of this resemblance, we can never gain a perfect knowledge. —

We have now arrived at some results important for what is to follow. We have seen that it is really possible for human beings to construct a perfect science as the mathematics indicated above, and we have further seen that the condition is, that all the objects dealt with are our own mental objects, fully determined by our definitions. The definitions here always precede the objects and thus create them. As soon as we turn to the real world, we have to commence with the objects. The definitions come in the second line and are never to be taken as exhaustive, for which reason we must here always remain at the approximation and the probability. Therefore geometry is not a science dealing with objective space and the like. With regard to such a space we know hardly anything, not even if such a space exists. A discipline relating to this would moreover be an empirical science, a chapter of geography, astronomy or even metaphysics with all the imperfections of an empirical science. Nor do we find any section dealing with such an objective space in the mathematical treatises. We learn in mathematics that, in accordance with our human concept of space, we can imagine points, lines, surfaces and bodies, and find necessary relations between the configurations formed more or less arbitrarily from them. Many of these configurations are evidently constructed as a kind of copies from the observed real objects. But they are then always simplified or idealized so that perfect treatment of them becomes possible.

III.

The above mentioned conception of mathematics as a formal, and thereby in a certain sense perfect, science closely accords with an important fundamental theorem set up already by PLATO. To a perfect science — he says — must correspond perfect objects. In accordance with this he declared that as regards the changeable things in the world of sense only more or less uncertain suppositions can be formed. If a perfect science like the true philosophy, „Dialectics" is to be possible, a system of perfect, unchangeable objects must be found, and thus he was led to the assumption of his Ideal World, a world of unchangeable models, after which the objects in the world of sense were formed as imperfect copies. ARISTOTLE rejected this theory and asserted that PLATO had only needlessly redoubled the things in the world. A whole school of philosophers in our day has however again reproached ARISTOTLE with having essentially misunderstood PLATO, and maintained that PLATO's ideas were not a number of independent things in an objective ideal world, but only a system of ideal images or pictures in the human mind.

It will hardly do, however, to interpret PLATO so freely. On the other hand it cannot be denied that, even if PLATO often speaks of his ideal world as if it were an objective collection of models, he has also several expressions indicating that the ideas as a collection of ideal pictures or ideal concepts in his mind have also been of significance to him. From the beginning the ideas had evidently arisen from the Socratic forming of concepts, and so far they had their proper home in the mental world itself, in which, according to PLATO, they grew up as by a kind of remembrance, and his conception of the nature of mathematics, which, as is well known, became of great importance to the Greek mathematicians, is especially in full accordance with these last reflections. PLATO evidently considered mathematics as a formal or ideal science, a science dealing not with the objects of the world of sense, not with the designed triangles or squares, but with the ideal pictures in the mind, with "the Idea of the triangle or square", which neither arises nor perishes, but is eternal and unchangeable. He therefore compared mathematics with dialectics which also treats of the ideal objects, and he gradually approached the two disciplines more and more to each other, while he did not at once place them on the same footing, as mathematics must buîld on assumptions which he did not think dialectics needed. In so far there is still in the beginning a certain vagueness about him, likewise appearing when he wants to place astronomy and music in the domain of ideas.

But 50 to 100 years later Euclid no doubt maintained exactly the same conception of mathematics. Unfortunately he did not open his Elements, which are otherwise rich in definitions, with a general determination of what is really to be understood by mathematics. If that had been the case the future with regard to mathematics would no doubt have been quite different. This very omission, in a work so anxious to strike bottom, seems however to indicate that the conception entertained by Plato was still quite unshaken, and that Euclid had the same conception may also be seen from different features in his work. First of all the decisive importance everywhere assigned to the carefully arranged definitions testifies to this. Furthermore unlimited straight lines, circles with any length of radius, etc. are considered. But for such things there was no room in the actual Greek world. We may therefore safely assume that also Euclid has seen a formal or ideal science in mathematics, a science not concerning the real world, but treating only of ideal objects produced by our own definitions.

How the conception of mathematics was changed later on and became more or less contradictory, the ensuing sections will show. Meanwhile we shall pause at Euclid in order to examine whether his statement really corresponds to the ideal aim he set himself, or whether there are actually in his treatise such decisive shortcomings as the New-Mathematicians have imputed to him.

IV.

As we might expect, the Elements commence with a series of definitions. The opus is divided into 13 books, and each book opens with only the definitions which, besides those previously given, are necessary for its understanding. As the above mentioned objections only relate to what is stated in the first book, it will be sufficient for us to examine this, which of course must also contain the most important of the suppositions. Of such Euclid gives, besides the Definitions themselves, two kinds, the so-called Postulates and certain General Assertions. Also these must therefore be examined. We commence with the Definitions.

By a definition we first of all understand, as is well known, the determination of a concept. Euclid knew this quite well. He names his definitions ὅροι (ὅρος = limit). The definition must so determine the concept that we clearly and distinctly understand what the question is about and what not, and the definition generally performs this by pointing out certain marks which must be found in each individual or case that is to come within the concept. Of course the definition must neither be too broad nor too narrow, it must be adequate, and its terms must be so plain and distinct that no misunderstanding is likely to occur. It cannot be demanded that every term employed should itself have been previously defined, as this would carry us into infinity, but fortunately many terms are in themselves so plain that every reasonable misunderstanding is excluded.

Euclid knew all this perfectly well and was carefully observant of it with one single exception, about which later on. He also knew that we can distinguish between the real definition and the mere nominal definition, i. e. in this case between the mathematical and the purely verbal definition. Under the heading Definitions he has in his first book 23 numbers, out of which the 18 are mathematical and the 5 only verbal definitions. He distinguishes between them by using the verb "is" in every real definition, while the verbal definitions always take the verb "is named". Thus in the definitions 9, 16, 20, 21 and 22[1]. It is important to emphasize this difference, the New-Mathematicians having sometimes made the general accusation against Euclid that his definitions are very often insufficient. If this were the case, his exposition would be mathematically very imperfect, for the sufficiency of the definitions is, as we have seen, a vital condition for the possibility of mathematics as an exact logical science. But the accusation is highly unjust, and only obtains an appearance of justness by quite overlooking the difference between the real and the verbal definition. That Euclid's mathematical definitions, excepting the one referred to above, are entirely sufficient, is already evident from the fact that all the ordinary expositions of geometry up to our own day on the whole use the very same expressions, without having found any need of completing them. And when f. inst. it has been

[1] J. L. Heiberg's edition. Translated into Danish by Miss Thyra Eibe.

objected, that it is not permissible to define a square as a quadrangle with equal sides and four right angles, without at the same time proving the possibility of such a figure, then it is overlooked, partly that this and some few similar definitions plainly appear as quite preliminary verbal definitions, and partly that the possibility of the corresponding mathematical objects is always demonstrated before they are introduced in the system. EUCLID has thus only allowed himself to explain a couple of names, shortly before dealing with the corresponding mathematical objects.

The before mentioned single unhappy definition is the definition of an angle. This is really erroneous, judged by modern standards. Professor ZEUTHEN has shown that this is due to the fact that in EUCLID's day the conception of an angle was still rather undeveloped among the Greeks. On this point, then, EUCLID was unable to rise above his contemporaries. But on the other hand it must be remarked that in return he deals so cautiously with the imperfect notion, that he introduces no errors in using it. If in the modern way he had defined an angle as the difference of direction between two straight lines issuing from the same point, measured in revolutions or parts of a revolution, his system would have remained quite unchanged. With regard to the question here discussed, the said imperfection is thus of no consequence. —

But the definition ought in certain cases to be something more than the mere determination of a concept. We define a horse as a single-hoofed quadruped with such and such marks, but we do not add that such an animal really exists. For it is here, as often, the case, that the existence of the thing in question is certain and well known, while it is only the limitation we are in search of. On the contrary it might not be so suitable to define: The dronte is a species of pigeon etc. or a mermaid is a female being etc. It is more proper to say: The dronte was a species of pigeon . . . , or is an extinct species of pigeon . . . , a mermaid is a fictitious being etc. In other words, wherever there may be any question about existence of one kind or the other, the definition should also be a guide on this point.

Now the mathematical objects are, as has been shown, mental objects. Whether perfectly straight lines, circles, ellipses, conchoides, tractrices etc. are to be found in the real world or not, does not in the least concern the mathematician. Mathematically it is only required that for each of the definitions a corresponding object can be formed without contradiction in the mind. The definition is in itself only a series of words. I must therefore be able to transform this series of words into a mental picture, and this transformation must be quite exact.

As a rule, however, this process is very easily performed. If I sketch roughly a triangle on the blackboard, saying: let this represent a rectangular plane triangle, then the ordinary twelve year old boy will at once in his mind form a picture of the mathematical triangle, and just as easily he proceeds from the wording of any plain definition to the mental picture. Sometimes the definition itself will point out the road which he has to follow in order to effect the transformation as easily as possible, informing him how the demanded object arises out of a simpler one: A cylinder is the body produced by a rectangle turning about one of its sides, a cone is . . . . The definition is then called genetic. Nor will this case present any difficulty worth mentioning, to the beginner or the less intelligent student.

I am quite convinced that if we, keeping in view all the above named demands, were more fully to examine the 17 Euclidian definitions still remaining, we should not be able to point out any justifiable objections whatever. All the objections made in course of time by the New-Mathematicians will no doubt appear to the impartial observer artificial and unjust, posited principally with the aim of safeguarding the new mathematics. Fortunately they are all directed against some few of the definitions, for which reason it will suffice to examine these.

All the principal attacks are primarily directed against Euclid's definition of the straight line.

In the dialogue Parmenides will be found a definition of the straight line, which somewhat freely translated would read: A straight line is a line which becomes a point when looked at from one end. This definition has at least the advantage of being very perspicuous. It is, as will be seen, just this definition every artisan, nay, every practical man will use: You look along the edge that ought to be straight, and examine whether the middle is covered by the ends, as it is expressed in the dialogue.

Yet this definition will not do. For it only informs us that the straight line follows the line of vision, but of the nature of the latter we are not informed. One would, however, be inclined to believe that Euclid had this definition in mind when forming his own. He avoids the shortcoming and says: "A straight line is that which lies evenly between its points." Thus Miss Eibe's translation. Euclid himself says that it everywhere lies ἐξ ἴσου, i. e. „in the same manner", between its points. But whether one or the other of these expressions is used, it is surely impossible to find just objections to this definition. It obviously compels us of full necessity and with unanimity of interpretation to form just that mental picture which by all impartial people is called the mathematical straight line.

The straight line must everywhere lie in the same manner between its points. By this the broken and the curved line are excluded, just as every little wave is excluded. I cannot try to unite these properties with the definition without conscious contradiction. The direction of the line must everywhere be the same; the straight line is the line with only one direction. It is everywhere the direct way from any one of its points to any other; it is therefore fully determined by two points, and it does not alter its position in space by turning round these. Nay, without going too far we may add, that it is the shortest way between two points. Against this definition which is sometimes used, Helmholtz has urged the objection, that in that case the definition must either presuppose the use of a measuring rule or only represent a jugdment by the eye. This objection is however unjust. Every savage, who has never seen or thought of a rule of measurement, not only knows that the direct road is the shortest but will also involuntarily comprehend that such must necessarily be the case.

The various definitions given above are those which mathematicians have used in the course of time. It will be seen that they all agree exactly with the definition given by Euclid. With insignificant verbal differences they all denote one and the same thing, and they all with unanimity and of necessity call forth exactly the same picture in the mind. No one, unacquainted with the succeeding historical evolution of mathematics, will be able to understand what might here be objected.

Euclid might even have said without any harm: The straight line I do not define, for what a straight line means, everybody knows so well that a definition would scarcely bring him any elucidation.

The straight line is the first and simplest definite conception with regard to space in the human mind. The point we first grasp as the intersecting point of two lines or as the initial or final point of a line. The straight line is such a fundamental determination in the human concept of space that, in fact, all succeeding concepts issue directly or indirectly from it: the plane, the angle, the circle, in short the whole geometrical system, the parallel-postulate, as will be shown, not excluded. To exclude this concept would mean annihilation of all true mathematics.

The various objections to the definition of the straight line are however easily dismissed. It has been said that the definition was not "geometrical" enough. But what does this vague expression mean? It might even be unfortunate if the first essential definition itself were „geometrical". It must be sufficient that it fully does its duty. It has been said, that the definition does not determine the nature of the line, but only mentions some of its properties. But normally we always define by indicating marks or properties. It has been said, that Euclid does not use the term "direction" at all. But that does not matter, as by the expressions used he evidently determines everything regarding the direction of the line. Nay, it has even been stated as an objection, that it was in reality an impossibility for human beings to imagine an exactly straight line. But just as this statement is hardly to the point it is also in itself quite wrong. We form our mathematical ideals in quite the same way as we form our other ideals: by eliminating all the imperfections. I can see the nearly straight pencil line or the nearly straight taut string, but I know that neither the line nor the string is quite without varying thickness or direction. All these visible and possible deficiencies ought not to be taken into account, I say to myself; and this resolution in connection with the still imperfect perception thus constitute what I call the mental picture or image of the straight line.

Regarding Euclid's definition of a plane superficies we can be very brief. It is, as might be expected, quite analogous to that of the straight line, and the objections very nearly correspond. We may therefore at once proceed to the succeeding assumptions.

V.

The next group is formed by the so-called Postulates, ἀιτήματα, five propositions of which the three first say: Let it be granted that a straight line may be drawn from any one point to any other point, that a terminated straight line may be produced to any length in a straight line, and that a circle may be described from any centre and with any radius. The fifth proposition is that very parallel-postulate mentioned before, which Euclid might well have formulated in close conformity with the three first ones, saying f. inst.: Let it be granted that from a side and two angles at that side, being together less than two right angles, we can always construct a triangle.

In contradistinction to these four propositions, of which each demands a certain construction granted as mathematically possible, the fourth says: Let it be granted that all right angles are equal to one another.

This last pronouncement must certainly be taken as a necessary supplement to the insufficient definition of the angle, which does not at all determine how the angle is to be measured. With this postulate we need not, therefore, occupy ourselves any further.

The four other postulates must doubtless be taken as expressions of the first immediately certain consequences of the definitions, that is, as a kind of theorems so simple that EUCLID has thought it unnecessary to give special demonstrations of them. It would hardly be right to take them as necessary supplements to the definitions. Neither the definition of the straight line nor that of the circle needs the least assistance, and which definition would the fifth postulate help? EUCLID has unquestionably taken also the fifth postulate as an immediate consequence of the nature of the straight line as determined by the definition, and in this respect a great majority will surely agree with him. On the other hand it must be admitted that this postulate implies a somewhat greater immediate step than the three first. It is therefore easily understood that a certain desire would arise to get just this postulate transformed to an ordinary, properly demonstrated, theorem. Only, why this could not easily be done, why more than two thousand years were to pass in indecisive attempts, and after this a whole revolution was to follow, these are things which we still do not understand.

Before we pursue this matter any further we have however to examine the third kind of assumptions. EUCLID calls them κοιναι ἔννοιαι. Miss EIBE translates this as General Concepts. They are however not concepts but judgments, assertions; it would therefore be more correct to say General Assumptions or the like. They are the following five propositions: Quantities which are equal to the same are equal to one another; if equals be added to equals the sums are equal; if equals be taken from equals, the remainders are equal; magnitudes which coincide with one another are equal to one another; the whole is greater than its part.

Euclid puts these general assumptions last and evidently does not treat them with special interest. He defines none of the expressions used, and the propositions are purely analytical propositions, i. e. tautologies. That the whole is greater than its part is a matter of course, as we only use the word "part" where there is still something wanting.

The addition of these assumptions, therefore, does not further enlighten us, and in several modern text-books they have also been left out without this causing the least inconvenience.

Several older editions of the Elements call this section Axioms. This is very unfortunate, as by an axiom we always understand a positive, synthetic assertion; no doubt the said heading has only been occasioned by the fact, that several really synthetic assertions were formerly included in this section; thus f. inst. the fifth postulate was for some time set up as the eleventh axiom.

Sometimes an unfortunate course has been taken in connection with this section, a course which has specially asserted itself in romantic philosophy, but has also shown itself in the new mathematics arising at the same period. The romantic philosophers often found it too troublesome to define every concept they introduced; so they said: Just read through the whole book, then you have got everything defined. In a similar way the new-mathematician F. Schur says (in his Grundlagen der Geometrie p. 3): We will not define, and D. Hilbert says (in his Grundlagen der Geometrie

3. ed. p. 4): The Axioms of Arrangement define the term "between". The truth is, however, this, that if I do not know beforehand what "between" means, I shall understand nothing of what is said about the Axioms of Arrangement. And with just as little right may we say that EUCLID's General Assumptions define the terms: Quantity, Magnitude, Equality, Congruence etc.

VI.

By this we have arrived at rather a surprising result: EUCLID, according to the generally adopted, latest, and best edition of the Elements, has set up no axioms at all.

This must not, however, dismay us. Many an excellent modern presentation does not mention any either. Definitions we must have, for it is only through such that the mathematical objects are created, and in every exact science it is very desirable always to know precisely what the question is about. Postulates, in the meaning of immediately evident consequences of the definitions, will also necessarily appear. But now the Axioms? What is a mathematical axiom?

It might be said: A mathematical axiom is a necessary assumption directly following from the human conception of space, the denial of which would cause contradiction in the mind. But in so far we need only proceed with sufficient caution, take only necessary steps, and beware of all self-contradiction. Of course all mathematics must thus rely on axioms, even if they are not directly pronounced.

Here it might be remarked that it would be of great interest to collect, arrange, and formulate all the necessary axioms. Certainly! But this would for many reasons be an exceedingly difficult task. HILBERT's arrangement is, as may easily be seen, quite superficial and dogmatic. It is not derived from any single principle, and we have no warrant for its sufficiency or correctness on the whole.

Somebody might perhaps interject the remark: But it would be sufficient to posit a certain number of axioms, and then proceed quite logically from them. By this we should at once obtain one mathematical system.

To this proposal it must, however, be objected: It will not do thus to confound axioms, definitions and objects. In the case of the mathematical objects and their definitions such a proceeding would be allowable. One mathematician may restrict himself to triangles, quadrangles, polygons and circles, the other may deal with ellipses, parabolas and hyperbolas etc. etc. Thus, as is well known, each mathematician often works in his own particular field. But these fields are all supplementary sections of the entire human mathematics. To treat the axioms in the same way would be quite another thing and perhaps incorrect. It would at all events be necessary carefully to examine a few cognate questions first. It might f. inst. be possible that, by thus parcelling out the demands of the human conception of space in various more or less arbitrary groups, we might be led to form a diversity of quite arbitrary systems, of which each at best would be a series of connected mistakes, while one of these systems might be opposed to the other and none of them therefore be of the least scientific value, as no reasonable choice between them might be possible.

On the other hand we might also suppose that the human conception of space was in its inmost nature such a unity, and all its special demands so closely united, that in building up from a part of them we should inevitably be led to observe the others. Thus ordinary geometry has in fact always observed a far greater accordance with the nature of our conception of space, i. e. maintained far more axioms than were ever formally set up. In so far no other inconvenience might result from this proceeding than the vagueness, that we imagined we had the complete foundation in the posited axioms, while in reality we only had some of the links.

And we should moreover incur a serious danger. For it might easily be conceived that in arranging the limited system of presuppositions we might confound some non-knowledge with some positive knowledge of the opposite kind, thus introducing contradiction and misstatement. Suppose f. inst. that we did not rely on Euclid's parallel-postulate. We must then also give up the proposition, that through a point outside a straight line can always be drawn one, and only one, parallel to the line. But this ignorance of ours as to whether just one parallel can be drawn, we unfortunately express by the positive assertion, that of parallel lines several can be drawn, and thus we have at once the possibility that a positive error has been introduced into the system. But such confusion of non-knowledge with positive opposite knowledge is not at all uncommon.

Thus we find various circumstances warning us against the proposed limitations of axioms, and our scruples increase as soon as we examine the psychical agencies which form mathematics.

With regard to these activities, however, we find rather vague and contradictory opinions in most mathematicians. Most of them no doubt feel inclined to call exact mathematics a pure logical science, thus accounting for its perfectly certain and exact results. Such must arise from human thought, while experience and the senses can only give us approximations.

But already here differences of opinion appear, as some will look to "thought alone" while several others maintain that also intuition is a necessary co-operating factor, not in the sense that it is allowable to make great compound steps by intuition alone, but that the last and least steps must always be made by its aid. And this, again, is understood now in one way and now in another: Some think that these intuitive steps are to be found everywhere, while others assume that certain mathematical disciplines as f. inst. arithmetic and algebra are obtained by "pure thought" alone, for which reason they are sometimes termed pure mathematics, while on the contrary geometry is interpreted as pure mathematics, applied to space, and therefore specially needing the aid of intuition, according to some, everywhere, according to others only in its opening chapters, where the definitions, axioms etc. are given.

The concepts here used are, however, shallow. If the origin of mathematics is really to be understood then we must proceed in a more precise psychological way. First of all we must take note that we cannot thus place thought and intuition opposite to each other, as if they were two mutually independent activities, of which each could manifest itself separately. So-called "pure thought" taken in the sense of thinking without intuition, does not exist at all. It is a fictitious invention by the romantic philosophers. To think is to deal with perceptions, and even if these are sometimes only perceptions of words, the intuition is already there. To think is in a broader sense to judge, in a narrower sense to conclude, to reason. But already in the plainest judgment as: two and two make four, there is plenty of intuition, and even for the most abstract judgment the same holds good. It is therefore clear that in all mathematics, and everywhere in mathematics, intuition takes its part. Exact mathematics may justly be called a pure logical science, but thereby intuition is not at all excluded; it is present in all thinking, also in logic itself, and it plays in fact a far more important part there than is generally supposed. Take a simple logical fundamental truth as f. inst.: If *A* is *B* and *B* is *C*, then *A* is *C*, i. e.: If a ball is in a box and the box in a drawer, then the ball is in the drawer. Why are we so absolutely sure of this? And why do we deny that if *A* is the son of *B*, and *B* the son of *C*, then *A* is the son of *C*? Every one will comprehend that it is only possible to distinguish between what is true and what is not true by transforming the words into pictures, and f. inst. discover, that if we imagine the ball in the box and the box in the drawer, then we shall be unable at the same time to fancy the ball outside the drawer without getting our consciousness split. All thinking is ultimately thinking in pictures. If you ask me: What is 7×17?, I may perhaps alone by hearing the words themselves be able to answer: 119. But then I have not actually thought, but only remembered. If I am really to think then I must pass by the sounds, the pictures of words, and cling to the pictures of things. Only these compel me to the definite result. Thus all necessity of thought arises from necessity of intuition. Thinking without intuition is the dark, in which all cats are grey. —

There is therefore no essential difference between arithmetic and algebra on one side and geometry on the other. Just as arithmetic and algebra arose among the Greeks as a part of geometry, thus they are still a kind of geometry, and like this, based ultimately on the human conception of space. Only, in modern times this is veiled in a high degree by the extensive and excellent language of signs which has gradually been introduced.

But regarding human thought, we must make one more remark. It is in a peculiar way a unity. It is the custom in Logic to give a plurality of so-called logical principles in order to determine its fundamental nature. But in reality all these principles are only different expressions of one and the same proposition, the so-called principium identitatis, and thinking may therefore simply be defined as a dealing with perceptions, obeying the principle of identity, and with cognition for its aim. Every breach of the laws of thinking may be represented as a contradiction, and the principle of contradiction, which excludes contradictions, is simply the indirect expression of the principle of indentity.

While we are thus unable to separate intuition from thinking, as thinking without intuition is simply impossible, we may on the other hand maintain, that just as intuition is the producing, thus the principle of identity is the controlling element in mathematics. Now human intuition may, as is well known, err. When mathematics, in spite of the great part intuition takes in it, can yet keep its character as a perfectly reliable and exact science, this is due to the fact that in mathematics we always dissolve each greater and therefore more uncertain step of intuition in a plurality of lesser and more certain ones, and in the domain of our conception of space this will always be possible to such a degree that these last steps, which all contain only rough and never minute valuations, become of an absolute certainty, just as certain as the fact that two and two make four, or that a horse is different from a butterfly.

But all these final petty steps constitute exactly the real mathematical axioms, the number of which in so far is pretty unlimited. An exhaustive enumeration would therefore, as said, be rather difficult. On the other hand we may indicate the conditions which every axiom has to comply with: It must be an immediate absolutely compelling step of intuition. We may likewise easily bound the whole system of axioms: It must contain an acknowledgment of each of the demands of the human conception of space, i. e. every demand, the denial of which would beyond doubt give contradiction. Therefore it will not do to imagine phantoms like spaces of four or more dimensions. For I may quite well form this sensation of sounds, but I cannot form any corresponding conception of things. In my conception the three dimensions are the complete extension, and, owing to the near connection between thought and intuition, I must therefore doubt that the fundamental laws of my thought would hold where such a situation really existed. In case of a fourth dimension I should lack every security that, in the case mentioned above, I had reasoned justly regarding the ball. I should have given up my faculty of discrimination, and could not therefore logically proceed.

And just as little would it be possible for me f. inst. to form a concept of space without free mobility for the contained objects. Obstructed motion calls forth the concept of real impediments, and such lie outside the concept of space. We may therefore in geometry be quite justified in using displacement, turning, and reversing of the figures in all the ways which our conception of space allows. Every mechanical removal, that might alter relations, is of course out of the question with regard to the ideal objects, but to exclude a removal in thought would be a contradiction. And just in the same way it must be possible to imagine both straight and curved lines, unlimited as well as limited, in accordance with the demands of our intuition. That mathematics deals only with our own mental objects and not with what might be found in a possibly objective space of real or fictitious formations, has already been said.

These hints may be sufficient to show what, in accordance with the aforesaid, ought to be understood by the mathematical system of axioms. As will be seen, it is just this — we might say unwritten — system which not only Euclid, but also the great majority of mathematicians who have attached themselves to his Elements, have, partly involuntarily and partly with full conciousness, followed.

VII.

While all the preceding remarks may be regarded as a series of truisms, we shall now go on to examine some positive occurences through which the conception of the nature of mathematics and thereby mathematics itself, and first of all geometry, involuntarily alters its character, thus opening the way for some misapprehensions finally resulting in the arrangement of the new mathematics.

Simultaneously with the gradual decay of Greek philosophy Plato's doctrine, so important to mathematics, concerning the ideality of the mathematical objects, was forgotten. And it was not a passing forgetfulness. The Scotch philosopher David Hume is the first who brings it to light again for a time. But at the same time geometry tended more and more towards being considered as a real science, a science about objective space and the different objective formations that were supposed to exist in the same. Thus geometry is conceived by SACCHERI, by LAMBERT, by GAUSS, by LOBATSCHEFSKY, by BOLYAI, by RIEMANN, by HELMHOLTZ, in short by all those who were indirect or direct co-operators in the founding of the new mathematics.[1]

That the important distinction between formal and real science was thus gradually lost sight of by the mathematicians, will surprise us all the less, when we consider that the same was the case among philosophers. As in the transition period between the renaissance and modern times Greek philosophy had in great measure decayed, while Greek mathematics still remained unshaken, philosophers recognised that they must make a fresh start, and while in England it was elected to adopt the empirical method of the rising natural sciences, all the continental philosophers found that they need only choose the method used in mathematics, and everything would be right. That mathematics creates its own objects while philosophy meets with its objects, and that the method suitable for the first discipline is therefore entirely unsuited to the second, occurred to nobody. SPINOZA wrote his Ethics in apparently pure geometrical form, WOLF wrote his whole system in the same manner, and most of the other continental philosophers did more or less the same thing.

At last HUME in his famous "Enquiry on Human Understanding" reiterates the Platonic sentence and declares mathematics to be an exact science as it only deals with the "relations of ideas" while he is sceptical towards all

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[1] ENGEL und STÄCKEL: Die Theorie der Parallellinien von EUCLID bis auf GAUSS.

ROBERTO BONOLA: La Geometria non euclidian; in German by H. LIEBMANN.

MORTON C. MOTT-SMITH: Metageometrische Raumtheorien. In this an ample list of books concerning this subject.

real sciences, claiming to treat "matters of fact". His Enquiry appeared in German translation in 1755, and influenced most likely by this, KANT expresses the same thoughts in his famous prizeessay from 1764: What else a cone may be, it is not easy to know, but in mathematics it is only that body which is produced when a rectangular triangle turns round one of its lesser sides. Thus in mathematics the objects always follow the definitions, therefore in mathematics we can obtain certainty. But in all real sciences the definitions follow the objects, therefore everything becomes uncertain here. — So clear-sighted was KANT in 1764. But when in 1781 he published his "Kritik der reinen Vernunft" he had forgotten HUME's distinction. Now geometry is the science dealing with space, and arithmetic the science dealing with "time or number", and with the object, among others, of retaining the absolute reliability of mathematics he makes space and time exclusively subjective forms of apprehension. If they were objective, the triangles in central Africa might be of quite a different nature to those in Königsberg.

But this first vagueness of the mathematicians inevitably causes some others. If geometry is a real science, a science relating to objective space, and if it deals with lines, surfaces etc., then these lines and surfaces are inevitably also made objective, and involuntarily they apprehend them as an army of mystical beings peopling space beside the quite real things as globes etc. In other words, it is that form of PLATO's doctrine of the ideas that ARISTOTLE rejected, which now rises from its grave, and is, half consciously, by the mathematicians restored to glory and honour in this "Astral-Geometry" as it is sometimes termed. It is according to this metaphysics that Gauss measures the triangle between Brocken, Hohehagen and Inselsberg in order to satisfy himself fully as to the total amount of the angles of the triangle; it is according to this that Lobatschefsky measures the parallaxes of stars, under the belief that they can give him information regarding the curvature of the objective straight line. That both undertakings are rather absurd is not at all obvious to them, as little as it is plain to them that, if geometry be a real science, its exactness, if we are to be consistent, will be a thing of the past.

And out of this second error arises yet a third. If the mathematical objects are not entirely created by ourselves, but mysterious beings, possessing some kind of independent existence somewhere in objective space, then of course we cannot define them with exactness. The mathematical definitions will then take on quite a new character. They become more insignificant; we can now never be sure that the definition is ideally exhaustive, i. e. of such a nature that all other properties of the objects are only the logical consequences of those given by the definitions. As in the real sciences, we must now always be prepared to meet unforseeable surprises, and Schur is really quite consistent when he thinks that it is not advisable to define at once, as the future might possibly overthrow the definition.

Of course it should not be expected that we can ever find a really consistent and complete application of the conception here indicated. In that case it would be too evident that it was no longer mathematics we had to deal with. If the Euclidian straight line is exchanged for some approximately straight reality or half-reality in space, it will evidently for many reasons never be possible to know, how this methaphysical object curves. A half-mathematical way is then employed, and certain simple curvatures selected to be dealt with. These objects are not, however, quite mathematical either, for it is then attempted — of course in vain — by actual measuring to find out the length of the radius of curvature in question.[1]

Already among the ancient Greeks the above mentioned neglect of the definitions begins to appear. The New-Platonist Proklos, who has commentated the first book of Euclid's Elements, tells us, that against the parallel-postulate the objection, amongst others, has been urged that it might be possible, that f. inst. a straight line together with a hyperbola or a conchoid could form interne angles with an intersecting straight line, together less than two right angles, without meeting, — an assumption which Euclid has never contested and would no doubt with great readiness have agreed to.

VIII.

As now the three last mentioned assumptions, in spite of their incompatibility with the very nature of exact mathematics, gradually gained more and more ground as the half involuntarily accepted base of mathematics and all mathematical investigation, it will not be difficult to understand how the endeavours of more than two thousand years to get the parallel-postulate demonstrated, could not but end apparently without decisive results, i. e. without having freed the Euclidian system from the alleged defects, and so having established its autocracy as a mathematical base.

How easily the parallel-postulate may be transformed into a theorem if we do not let ourselves be confused by the above mentioned errors, will first be shown.

[1] Vide Engel und Stäckel or Bonola.

We may reason thus: As it is the easiest thing on earth to imagine curved lines which in spite of the given conditions do not meet, it must be the straightness of the straight line which is the decisive point. Now if we take the definition in full earnest and remember that the mathematical straight line is nothing but our own creation, fully determined by our definition which has unmistakably excluded all curved lines and only retained the one, whose direction is everywhere one and the same, then the proof will follow quite easily thus (Fig. 1): Let the table represent a plane and l a straight line the direction of which we may call zero. Let m be another straight line which forms $\angle a$ with l and therefore has the direction a. Let n be a third straight line which forms $\angle b$ with m and therefore has the direction $a+b$, and let l again form $\angle c$ with n and therefore have the direction $a+b+c$. Now l has also the direction zero. Therefore $a+b+c$ must be just one turn or 4 right angles, and as the angles $a+\alpha+b+\beta+c+\gamma$ make 6 right angles, the three angles of the formed triangle, $\alpha+\beta+\gamma$, must make 2 right angles. But if the angles of any plane triangle are equal to two right angles, we can always of a side and the two angles at the side, these being less than two right angles, construct a plane triangle, and thus the postulate is demonstrated. If the angles are, on the contrary, together equal to two right angles, then a plane triangle can not be constructed, and thus EUCLID's parallel theorem (1,27 & 1,28) is demonstrated.

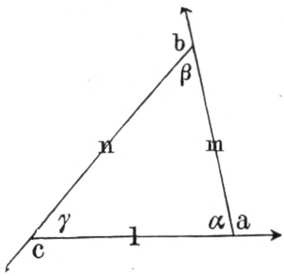

Fig. 1.

A similar demonstration has been given by the Danish astronomer Schumacher. He sent it to Gauss, but as he had omitted some intermediate links, Gauss found it insufficient.

Also the Danish mathematician Julius Petersen has given a similar demonstration. But just as he had finished it, he rejected it himself in virtue of two objections which in a characteristic way show the power of suggestion on the human mind; it had, as may be remembered, become a truism that the fifth postulate could not be demonstrated. Julius Petersen objects: But the demonstration is, however, of no value, for on a sphere it would not hold good. — Now we find on a sphere neither plane triangles nor straight lines, for which reason the demonstration cannot at all be applied here. The objection is therefore quite illogical. It is as if one would say: Denmark cannot after all have 3 million inhabitants, for this number does not hold good for England.

His second objection is that the three angles a, b, c, do not issue from the same point but each from a separate one. This objection is, however, also unjustifiable, for the line l with the direction zero is a straight line and has therefore everywhere the same direction. The line m therefore gets the direction a from whatever point of l it may issue. And so forth.

The new-mathematicians may no doubt object to the demonstration that Euclid has nowhere stated that the straight line is the line with but one direction, or that an angle is the difference of directions between two intersecting straight lines. But to this the reply must be that he really has said both, only expressed it in other just as definite words, and it is not his words but his thoughts, his system, we are disputing about. The following will show that it is quite a different kind of objections the new-mathematicians really want to put forward. If EUCLID had only used a single infelicitious or insufficient word, the natural proceeding would have been: straightway to put in a better term instead of the objectionable one. The new-mathematicians themselves proceed in this way with regard to EUCLID's definition of an angle in which he had in fact used undesirable expressions. But such an incidental lapse ought not to cause the condemnation of a whole system.

Before proceeding we must mention a third Danish mathematician who has given a completely exhaustive demonstration of the validity of the parallel-postulate. It is C. RAMUS who was not only a mathematician of uncommon penetration, but moreover specially interested in rendering the exposition of mathematics as logically incontestable as possible. His starting-point is that plane triangles with a side and two adjacent angles separately equal, are congruent. From this it follows that the side and the two angles determine the two other sides and the third angle, and as angular size can be measured in pure numbers in relation to the right angle, whereas the length of straight lines must always be measured in relation to quite an arbitrary piece of line, the two angles of the triangle must already determine the third, quite independently of the length of the sides. But if now in a rectangular plane triangle, ABC, we draw the height from the right angle C, thereby dividing C in C_1 and C_2, we easily find, according to the before said, that f. inst. $C_1 = B$, $C_2 = A$, therefore $A+B = 90°$, and thus the sum of the angles of each of the three right-angled triangles = 2 right angles. From this again we easily get that the sum of the angles in any plane triangle must be two right angles, and thereby, as we have seen, the parallel-postulate is demonstrated.

The new-mathematicians will probably object to this demonstration, that the parallel-postulate is here only exchanged for another one, namely the postulate that similar plane triangles are possible. This objection is however unjustifiable. The author does not presuppose the possibility of similar plane triangles, but by his enunciations regarding the measuring of angles and lines he, on the contrary, proves that similar plane triangles can be constructed, a truth which, by the by, is already the necessary consequence of the concepts of straight line and angle, provided we do not include curved lines in the class of the straight ones.

It is therefore a not very happy expedient some of the opponents of the new mathematics have invented, in proposing that instead of the intricate parallel-postulate we might put the plainer axiom that similarity exists. Even more reasonable it would evidently be to set up the still simpler declaration of Pascal: Definitions must be respected.

Also the well-known English mathematician Wallis and the likewise well-known French mathematician Legendre have — besides several others — given demonstrations of the correctness of the parallel-postulate, which, granting the here given conception of mathematics, must be considered decisive.

It will, however, be unnecessary to examine closer all these particulars. Already from what has been said we may understand that if no agreement was obtained regarding the demonstrableness of the parallel-postulate, it was because so many mathematicians involuntarily took the straight line as a mystical object which it would not do to regard as fully determined by the given definition, which again occasioned the assumption that the definition itself was vague. In the above mentioned book by ENGEL & STÄCKEL it may be seen, how, as a rule, they, only timidly and only in the utmost emergency, venture upon a meek appeal to the definition.

IX.

Two attempts at demonstrations must, however, still be examined as they in no small degree conduced to the rise of the new mathematics. They are from the hand of two mathematicians, the Italian SACCHERI (1667—1733) and the Swiss LAMBERT (1728—1777). They both chose one and the same peculiar proceeding, and were thus rather unintentionally led to bring forward several of the theorems of the new-mathematicians. Both were themselves quite convinced of the correctness of the Euclidian postulate, just as both had no doubt that in their works they had given a decisive proof of its correctness. But both were unfortunately also to a certain degree subject to the prevalent vagueness about the nature of mathematics and the importance of the definitions, and they were therefore unable to maintain with sufficient power and acuteness that their essays in reality contained the exhaustive proof which they thought they had given. They both arrive at the result that a denial of the postulate would be in contradiction to the nature of the straight line, but what this really signifies they are rather uncertain about. SACCHERI's essay was published in the year of his death 1733; that of LAMBERT was written in 1766, but was only published in 1786, 9 years after his death, no doubt because he was not quite satisfied with it himself.

Each of the two essays has about the size of an ordinary school geometry. We must therefore restrict ourselves to giving a short report of the mode of proceeding common to both and the results obtained. Both essays are reprinted in Engel and Stäckel.

At two arbitrary points of a straight line, A and B, (Fig. 2), are drawn two equal straight lines in the same plane perpendicular to the given line, the ends of which are joined by a straight line CD. If AB and CD are bisected and a straight line EF is drawn through the bisecting points, the figure is divided in two congruent halves, and we shall have right angles at A and B and round E and F, and the angles C and D will be equal. But whether they are right, obtuse or acute we do not know as yet. We therefore make the three corresponding hypotheses, and using Euclid's 28 first propositions, which are not based on the postulate, we investigate what each of the hypotheses may teach us about the postulate.

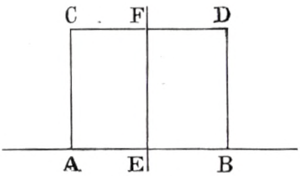

Fig. 2.

Both authors then find, that if C and D are supposed right angles, the sum of the angles in the plane triangle will also be two right angles, by which the correctness of the parallel-postulate is demonstrated.

If C and D are supposed obtuse angles, the sum of the angles in the plane triangle will be more than two right angles, and also from this follows the correctness of the postulate, as may easily be seen.

If finally C and D are supposed to be acute angles, the sum of the angles in the plane triangle will, on the contrary, be less than two right angles, and then the parallel postulate is incorrect, as the two lines will then only meet in certain cases.

Both authors further find, however, that while the first supposition does not involve the least difficulty, the second and third will produce the curious result that the propositions employed will force us to ascribe to the lines containing the two angles such lengths and forms that it will no more be possible to call them straight lines. *CD* f. inst., in case of the hypothesis of the obtuse angle, becomes curved with the concavity turned towards *AB*, and in case of the hypothesis of the acute angle, curved in the opposite direction. But as *CD* was given as a straight line, such contradictions have appeared that the two last hypotheses must be rejected, and the parallel-postulate again stands absolutely demonstrated. —

At first sight it looks rather astonishing that these two authors who more than any others have spent time and intelligence in securing Euclidian geometry, and have written so largely in favour of the parallel-postulate, should none the less by these very writings have become, we might even say — and it has actually often been said — the first exponents of the new mathematics, the very aim of which was partly or entirely to supersede Euclid. The riddle is not, however, difficult to solve. It all arises from the fact that there is something rather illogical in their proceeding.

For the parallel-postulate is not — as so often thought — a kind of fifth wheel to the geometrical coach, a kind of appendix which may be taken up or dropped at will, but it is, as we have seen, already latent in the very first and simplest geometrical object, in the concept of the straight line, taken strictly in accordance with the definition and in accordance with the whole human conception of space, taken in only one sense and not ambiguously, so that the name might also denote the curved line or perhaps even that only. If that is the case, the parallel-postulate is of course not included; that curved lines do not always necessarily comply with its demands is sufficiently evident, and obviously in good accordance with the opinions of Euclid.

But from this it may be seen that if an essay on the parallel-postulate is to be any good, the author must first of all have his own definite conception of that object: the straight line. He may either be convinced of the fundamental importance of the definitions in mathematics, and with Pascal maintain that they ought to be strictly respected; or he may think according to Schur that it is undesirable or dangerous to define at once, and then the objects will from the beginning stand in a somewhat dim light, just as the objects of real science. Both these points of departure at any rate possess a certain consistency.

But unfortunately the two authors both stand about halfway between these two points of view. Lambert, who is the least bewildered, says regarding the definition of the straight line: *Et hoc si dederis, danda sunt omnia.*[1] But in return he sees in all the definitions only hypotheses which may apparently at will be admitted or rejected. Probably he imagines behind these definitions a kind of real or half real objects in space, about the precise qualities of which only the surveyor, the natural philosopher, the astronomer, or the metaphysician can give us final information. Already this has, however, led us in a circle. For most of them, the surveyor, the natural philosopher, the astronomer, proceed

[1] Engel and Stäckel, p. 142.

by the aid of mathematics, and in mathematics we at once commence to work with the still hypothetical objects.

It is, however, only one aspect of LAMBERT's starting point we have here indicated. For on the other hand he discloses no little confidence in the more than hypothetical value of the mathematical definitions, and like SACCHERI he concludes by assuming his problem solved, because the lines at which he arrives are contradictory to the line determined by the mathematical definition.

This is vague. And of vagueness and illogicalness we find still more in the proceedings of the two authors.

If we start from the straight line as straight, it is, as we have seen, very easy to demonstrate the correctness of the parallel-postulate, and just as easily we then find the value of each of the above mentioned three hypotheses. Let the size of each of the two angles C and D be $(90+x)^\circ$, and let CD have the direction zero, then DB has f. inst. the direction $(90-x)^\circ$, BA the direction $(180-x)^\circ$, AC the direction $(270-x)^\circ$ and CD the direction $(360-2x)^\circ$, from which we see, that x must be $=0$ and the two last hypotheses therefore impossible. There is therefore no sense in going to them about the parallel-postulate.

But how do the two authors proceed?

They seek information about the parallel-postulate. To that end they set up the three hypotheses and examine them, using the 28 first propositions in EUCLID. This course of procedure contains the following errors:

They are not quite sure what a straight line really means. But if the Euclidian straight line is not unconditionally accepted, then it will not do to appeal to any single one of EUCLID's propositions, not even to the first 28. For, so to speak every one of them, nay even his postulates, become either quite indecisive, unmeaning or positively wrong, if the straight line is not exclusively the line he has thought of and defined as straight.

Suppose, however, that the two authors after all have been bent on basing on that very, Euclidian, straight line.

Then their mistake is that they have overlooked that an angle cannot at the same time have three sizes, but only one. Only one of the hypotheses can therefore be valid, while two of them can at once be declared impossible and thus unfit to disclose anything reasonable about the postulate. The proper proceeding would therefore have been: First to find out which of the hypotheses we might reasonably interrogate. But the two authors unconcernedly consult all three Euclidically, thus inevitably assuming, that they can also examine the impossible hypotheses by means of the Euclidian propositions. It may further be added that the authors of course ought to have stopped at once on perceiving that the lines showed themselves imcompatible with the concept of the straight line. They, however, proceed much further. In spite of all their undeniable acuteness they have thus comitted several considerable errors, and at least a great part of their results thus lose their validity.

This judgment must be passed on the two essays from the Euclidian point of view. The essays, however, appear in quite another light if viewed from the starting point which was largely beginning to predominate in the period. The presuppositions are then the above mentioned: Mathematics is a kind of real science, the mathematical objects are a kind of real or half real entities which are not yet thoroughly known and therefore cannot either be defined beforehand with certainty. Whether the straight line is mathematically straight or a little curved, or perhaps sometimes the one and sometimes the other, is uncertain. We must therefore be prepared for any results.

But if the expression "straight line" contains such a diversity of significations, there will of course be no contradiction in advancing the three hypotheses as provisionally possible. We do not then know beforehand what kind of straight line we have in the figure. But it will certainly still be illogical to use EUCLID's first 28 propositions as a general test. This was, however, at first overlooked. That the lines of the figure sometimes unveiled themselves as straight lines, sometimes as concave, and sometimes as convex, created no doubts; such things might be expected. That after all a lot of "results" could be gained was taken as a sign that new and surprising discoveries were impending. SACCHERI's and LAMBERT's scruples about the incompatibility of the curved lines with the nature of the straight line was neglected. On the contrary, it was assumed that the investigations communicated had revealed to humanity that there were actually two kinds of geometry: The geometry of the right angle in which the parallel-postulate holds good, and the geometry of the acute angle in which it does not hold. The geometry of the obtuse angle was not set up yet, as here a certain difficulty had appeared. For also according to the hypothesis of the obtuse angle did the parallel-postulate, and thus the whole Euclidian geometry, prove to be in good order. But in the latter it was demonstrated that the sum of the angles of the rectangle is always equal to four right angles, and consequently the two obtuse angles must after all have been right angles. It was only later that it was discovered that the first 28 propositions of EUCLID could not all reasonably be used as tests of this hypothesis, and after this had been found out, the geometry of the obtuse angle was also set up as a justified new system. —

Lambert had, however, made some important discoveries, already touched upon by him at the end of his essay on the parallel-postulate, but whose far-going importance for the whole geometrical dispute he did not see or point out. Ingenious mathematician as, in spite of all, he was, he had always worked with a certain distrust and reluctance with the curved lines evolved from the original straight ones through the influence of the Euclidian propositions. He now finds that the results arising from the hypothesis of the obtuse angle are just those that are valid in spherical geometry; but it does not at the same time come home to him that all ambiguities and contradictions at once disappear and everything becomes clear and natural, as we perceive that by introducing the two obtuse angles we have driven ourselves out of the plane and simply developed a fragment of well-known spherical geometry under the impression that we were discovering something quite new, surprising, and almost incredible, in the domain of plane geometry. At the same time as the obtuse angles curved the lines, they of course also curved the plane, and for a spherical quadrangle between the equator, a couple of meridians and — not a parallel of latitude, but — another arch of a great circle, without the least subtlety we shall get every one of the results which before we had to place, against all our immediate insight, in the plane rectangle. In the spherical quadrangle the two meridians will bend towards each other, the two angles C and D will be more than two right angles, the arch of the great circle above will arch upwards, and yet be shorter than the base, and the spherical triangles will possess the proper excess beyond two right angles. The mystical assumption regarding the different kinds of straight lines resolves itself simply into the well-known truth, that just as in plane geometry we have straight lines thus in spherical geometry we have the great circles, the "straightest" lines on the sphere.

But this is not all. LAMBERT further finds that if we give the sphere an imaginary radius, we shall be able by ordinary algebraic-trigonometrical calculations to obtain just all those results we arrived at before by the hypothesis of the acute angle.

And finally he finds that in the domain of this imaginary spherical geometry, in the radius of the sphere, or in any function of it, we shall obtain a peculiar natural unit for measuring lengths, that is: just such a natural unit of length as we do not find in plane geometry, a fact by which, amongst others, the demonstration by RAMUS (p. 37) was made possible.[1]

By these discoveries of LAMBERT a great deal of the obscurity had passed away. But still clearer light was soon to come. At the commencement of the new century, GAUSS developed his General Theory of Surfaces, in which, amongst others, he set up his famous "Measure of Curvation" (Krümmungsmaass) for the curvation of surfaces in their different points, and showed — as later on MINDING — that for each such measure of curvation there is a particular geometry for the surface: If two surfaces have the same constant measure of curvation, they have also the same geometry. For they can then be placed exactly against each other without being stretched or constricted. The measure

[1] ENGEL und STÄCKEL, p. 199—203.

of curvation is the reciprocal value of the product of the two principal radii of curvation, $K = \frac{1}{\rho_1 \cdot \rho_2}$. Therefore plane geometry also holds good for the cylindrical surface, as the plane and the cylindrical surface both have the constant measure of curvation $K = 0$. For the sphere the measure of curvation is constant and positive: $K = \frac{1}{r^2}$, if r represents the radius of the sphere. For Lambert's imaginary sphere with radius ir it thus will be $K = \frac{1}{ir \cdot ir} = -\frac{1}{r^2}$. But the same constant negative measure of curvation therefore applies also to a surface which f. inst. is produced by the rotation round the X-axis of a curve having the product of its normal and radius of curvation constant and negative, as the said, opposite directed, lines become just the two radii of curvation for the surface. Such a curve is f. inst. the so-called Tractrix (Fig. 3). It is also characterized by the fact that its tangent has a constant length from its origin to its intersection with the axis of rotation, which is an

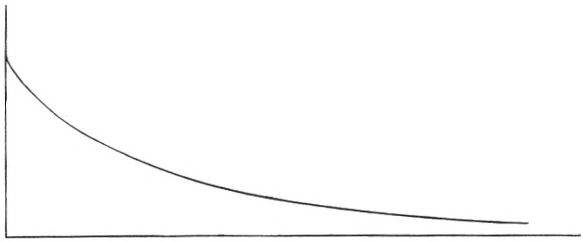

Fig. 3.

asymptote to the curve. The surface produced by the rotation is named a pseudo-spheric surface, and is formed somewhat like a flower glass with the sides bent somewhat outwards and with a stem infinitely long, tapering gradually down.[1] (Fig. 3).

Another form of pseudo-spheric surface arises by rotation of the curve given in Fig. 4, which also has the product of normal and radius of curvation constant and negative. It is a kind of redoubled contraction of the former one which however may also be completed symmetrically along the negative X-axis. The corresponding surface will, as may be seen, resemble the groove in the sheave of a pulley or the surface of a saddle.

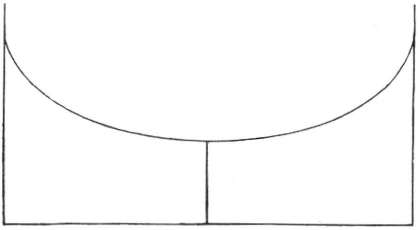

Fig. 4.

It will already be immediately evident how a quadrangle of straightest lines from equator of the last named surface will get the sides bent outwards and acute angles at the top line which is convex towards the equator, or of which the middle must be nearer equator than the ends, if it is to be the shortest way between C and D. The Italian mathematician BELTRAMI has shown the conformity between "the geometry of the acute angle" and that of the pseudo-spherical surface, but it will not after all be difficult even in a quite Euclidian way to find the results concerned, if we have regard to the before-mentioned circumstance that it

[1] The curve is represented by the equation

$$x = a \ln \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2},$$

where $a = y$ for $x = 0$ and $=$ the said length of the tangent. The measure of curvation for the surface will be $K = -\frac{1}{a^2}$.

is the same geometry as the spherical for a sphere with the radius ir. Some examples will illustrate the mode of proceeding.

Half the surface of an ordinary sphere is, as well known, $2\pi r^2$. The biangle with $\angle A$ (Fig. 5) has therefore the area $\frac{A}{\pi} \cdot 2\pi r^2 = 2 A r^2$. The biangle B will be $2 B r^2$, and the biangle C will be $2 C r^2$.

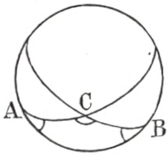

Fig. 5.

But from this follows

$$2\,(A+B+C)\,r^2 = 2\pi r^2 + 2\triangle ABC$$

therefore

$$\triangle ABC = (A+B+C-\pi)\,r^2,$$

and

$$A+B+C = \pi + \frac{\triangle ABC}{r^2}.$$

These two formulas give us the area and the sum of the angles for the ordinary spherical triangle. It will be seen that the sum of the angles exceeds two right angles proportionally to the area of the triangle.

We now imagine a sphere with radius ir and then from this get the two formulas

$$\triangle ABC = (\pi - A - B - C)\,r^2$$

and

$$A+B+C = \pi - \frac{\triangle ABC}{r^2}.$$

From this it will be seen that the sum of the angles in the pseudo-spheric triangle decreases from two right angles proportionally to the area of the triangle. If the sum of the angles $= 0$, which is quite possible in triangles with inward bent sides, the triangle will be the greatest possible here, and its area will be πr^2, equal to the area of an ordinary circle with radius r.

In order easily to get more results, we may use the well known hyperbolic functions

$$\text{Sin}\ x = \frac{\sin ix}{i} = \frac{e^x - e^{-x}}{2} = \frac{x}{\underline{|1}} + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \ldots$$

$$\text{Cos}\ x = \cos ix = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \ldots$$

On the sphere with radius r we have for the triangle ABC with the sides α, β, γ

$$\cos\frac{\alpha}{r} = \cos\frac{\beta}{r}\cos\frac{\gamma}{r} + \sin\frac{\beta}{r}\sin\frac{\gamma}{r}\cos A.$$

If radius be ir we therefore get

$$\cos\frac{\alpha}{ir} = \cos\frac{\beta}{ir}\cos\frac{\gamma}{ir} + \sin\frac{\beta}{ir}\sin\frac{\gamma}{ir}\cos A.$$

But now

$$\cos\frac{\alpha}{ir} = \cos\left(-i\frac{\alpha}{r}\right) = \cos\left(i\frac{\alpha}{r}\right) = \text{Cos}\,\frac{\alpha}{r},$$

$$\sin\frac{\beta}{ir} = \frac{1}{i}\,\frac{\sin\left(i\frac{\beta}{r}\right)}{i} = \frac{1}{i}\,\text{Sin}\,\frac{\beta}{r},$$

therefore

$$\text{Cos}\,\frac{\alpha}{r} = \text{Cos}\,\frac{\beta}{r}\,\text{Cos}\,\frac{\gamma}{r} - \text{Sin}\,\frac{\beta}{r}\,\text{Sin}\,\frac{\gamma}{r}\cos A.$$

Supposing now $A = B = C$, $\alpha = \beta = \gamma$, we get, as $\text{Cos}^2 x - \text{Sin}^2 x = 1$,

$$\text{Cos}\,\frac{\alpha}{r} = \text{Cos}^2\frac{\alpha}{r} - \text{Sin}^2\frac{\alpha}{r}\cos A,$$

$$\cos A = \frac{\text{Cos}\,\frac{\alpha}{r}\left(\text{Cos}\,\frac{\alpha}{r} - 1\right)}{\text{Cos}^2\frac{\alpha}{r} - 1} = \frac{\text{Cos}\,\frac{\alpha}{r}\left(\text{Cos}\,\frac{\alpha}{r} - 1\right)}{\left(\text{Cos}\,\frac{\alpha}{r} + 1\right)\left(\text{Cos}\,\frac{\alpha}{r} - 1\right)} =$$

$$= \frac{\mathrm{Cos}\,\frac{\alpha}{r}}{1+\mathrm{Cos}\,\frac{\alpha}{r}} = \frac{1+\frac{\left(\frac{\alpha}{r}\right)^2}{\underline{|2}}+\frac{\left(\frac{\alpha}{r}\right)^4}{\underline{|4}}+\ldots}{2+\frac{\left(\frac{\alpha}{r}\right)^2}{\underline{|2}}+\frac{\left(\frac{\alpha}{r}\right)^4}{\underline{|4}}+\ldots}.$$

But this fraction is $> \frac{1}{2}$, therefore $A < 60°$ and $A+B+C < 180°$. If however $\frac{\alpha}{r}$ decreases towards zero, the fraction will at the same time decrease towards $\frac{1}{2}$, and A will at the same time increase towards $60°$. We thus in this way get the same results as before.

Further we have on the sphere with radius r

$$\cos A = -\cos B \cos C + \sin B \sin C \cos \frac{\alpha}{r}$$

and if the radius is ir

$$\cos A = -\cos B \cos C + \sin B \sin C \,\mathrm{Cos}\,\frac{\alpha}{r}.$$

Now, putting $A = 0$ and $C = 90°$, we get

$$\sin B = \frac{1}{\mathrm{Cos}\,\frac{\alpha}{r}}.$$

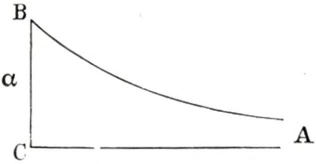

Fig. 6.

Here B is "the Parallel Angle for the distance α", i. e. the least angle that can be admitted in the distance α from the right angle C without the lines BA and CA intersecting.

If we put $A = 0$, $C = 90°$ and $B = 45°$, which may be done, the triangle getting the corresponding magnitude, we get from the last equation

$$\text{Cos}\,\frac{\alpha}{r} = \frac{1}{\sin 45^\circ} = \sqrt{2} = \frac{e^{\frac{\alpha}{r}} + e^{-\frac{\alpha}{r}}}{2} = \frac{e^{2\frac{\alpha}{r}} + 1}{2\,e^{\frac{\alpha}{r}}},$$

therefore

$$e^{2\frac{\alpha}{r}} - 2\sqrt{2}\;e^{\frac{\alpha}{r}} + 1 = 0;\; e^{\frac{\alpha}{r}} = \sqrt{2} \pm 1;\; \frac{\alpha}{r} = \ln(\sqrt{2} \pm 1),$$

where the lower sign must be rejected, as it would make α negative.

If the triangle is augmented with the one, symmetrical round α, we get a right-angled isosceles pseudo-spherical triangle with the two other angles $= 0$ and the height $\alpha = h$ from the right angle

$$h = r \ln(1 + \sqrt{2}).$$

This quantity is named SCHWEIKART's Constant, while

$$r = \frac{h}{\ln(1 + \sqrt{2})}$$

was taken by GAUSS as the Constant or natural unit of length which, according to LAMBERT, appeared in "the geometry of the acute angle". The Constant r is in other words the radius of curvation for the surmised straight lines in space.

While it is easy enough to find the ratio $\frac{h}{r}$, it has of course up to this been impossible to find h or r themselves.

LOBATSCHEFSKY however made several attempts. One of them has eben alluded to previously and may therefore here be considered a little more closely. It carries us back to Fig. 6 with $A = 0$ and $C = 90°$. From the obtained equation $\sin B = \frac{1}{\operatorname{Cos}\frac{\alpha}{r}}$ we further get

$$\frac{1}{\sin B} = \frac{e^{\frac{\alpha}{r}} + e^{-\frac{\alpha}{r}}}{2} = \frac{e^{2\frac{\alpha}{r}} + 1}{2\, e^{\frac{\alpha}{r}}}; \quad e^{2\frac{\alpha}{r}} - \frac{2}{\sin B} \cdot e^{\frac{\alpha}{r}} + 1 = 0;$$

$$e^{\frac{\alpha}{r}} = \frac{1}{\sin B} \pm \sqrt{\frac{1 - \sin^2 B}{\sin^2 B}} = \frac{1 \pm \cos B}{\sin B} = \begin{cases} \cot \frac{1}{2} B \\ \operatorname{tg} \frac{1}{2} B \end{cases}.$$

As $\frac{1}{2} B$ must be $< 45°$, therefore $\cot \frac{1}{2} B > 1$, $\operatorname{tg} \frac{1}{2} B < 1$, while $e^{\frac{\alpha}{r}}$ must be > 1, we evidently get

$$\frac{\alpha}{r} = \ln \cot \tfrac{1}{2} B; \quad -\frac{\alpha}{r} = \ln \operatorname{tg} \tfrac{1}{2} B,$$

$$\frac{r}{\alpha} = \frac{1}{\ln \cot \frac{1}{2} B}.$$

Supposing now — somewhat boldly — that C represents the sun, B the earth, α therefore the radius of the orbit of the earth and A an almost infinitely far off visible star, then, if the light proceeds along a straight line, $\frac{\pi}{2} - B$ will be A's nearly vanishingly small parallax. But if the path of the light is one of the supposed objective curved lines as AB, B may differ more from $\frac{\pi}{2}$, and if now the astronomical observations were to show that the so-called parallax $\frac{\pi}{2} - B$ is always perceptibly different from zero, we may assume that a great part of the deviation arises from the said curvation, and thus, through the just given formula we get an upper limit for the curvation, or a lower limit for the radius of curvation, r.

Unfortunately there is a good deal to be objected against this test, and amongst others it will be a decisive hindrance that the astronomers do not at all find the expected lower limit for the magnitude of the parallax. The astronomers (according to communications from the late Observator PECHÜLE) have measured parallaxes right down to 0,01 seconds of an arc, where the limit for thrustworthy measurement of the kind concerned lies.

If, however, we take double this parallax angle we get

$$\cot \tfrac{1}{2} B = \cot\ (45° — 0'',01) = \frac{1 + \operatorname{tg} 0'',01}{1 — \operatorname{tg} 0'',01}.$$

Taking the arc instead of the tangens of the very little angle, and remembering that the radius contains 206 264, 806 seconds we get

$$\cot \tfrac{1}{2} B = \frac{206\,264\,816}{206\,264\,796} = \frac{51\,566\,204}{51\,566\,199}$$

and thus

$$\frac{r}{\alpha} = \frac{1}{\ln \dfrac{51\,566\,204}{51\,566\,199}} = \frac{0{,}434\,2945}{\log \dfrac{51\,566\,204}{51\,566\,199}},$$

which very nearly gives

$$r = 10\,219\,000\ \alpha,$$

i. e. the radius of curvation, GAUSS's Constant more than ten millions of radii of the earth's orbit. For anybody who considers geometry an empirical (approximative) science about the real world, there is, so far, no special reason to set up a peculiar "the acute angle's system."

X.

The above given examples may sufficiently have shown us how fruitful were the ingenious remarks which Lambert so lightly threw out, but of which neither he himself, his contemporaries, nor the near future, saw the full reach. Not even after Gauss had published his famous measure of curvation, was it at once perceived that it was the simple and clear geometry of certain curved surfaces which had been considered. Of the mathematicians who on the whole expressed themselves regarding the two thousand year old controversy the majority no doubt subscribed to the assumption that at least one quite new geometrical system had really been discovered; and as there are many kinds of acute, but only one kind of right angles, the Euclidian geometry was now only considered as an extremely special, limited case which moreover had the defect that it used the highly contestable parallel-postulate as one of its presuppositions while the new system was based on altogether incontestable suppositions.

As Bolyai and Lobatschefsky, during the first half of the nineteenth century, independently of each other but both in close connection with the German evolution, published their detailed statements of the system of the acute angle, Lobatschefsky therefore named his principal work Pangeometry.

To dwell further on the two authors will be superfluous for the present purpose, as the statements of both are only a natural continuation of what had already been given by Saccheri, Lambert, Schweikart, Taurinus and others who had busied themselves with the three hypotheses. Only now it was unconditionally declared that it was a new and independent system that had been discovered. That the initial vagueness had in no way been removed is a matter of course. But is was partly concealed by the form of the exposition. Thus LOBATSCHEFSKY commences his Pangeometry by defining the plane as the locus of the intersection of equal spheres round two fixed points, and the straight line as the locus of the intersection of equal circles round two fixed points. These definitions would be sufficiently unique and Euclidian, if we had only been told that by a sphere and a circle LOBATSCHEFSKY understood the same thing as ordinary people do. But about this nothing is said. Both these concepts are themselves determined by the concept of the straight line. LOBATSCHEFSKY has thus only led us in a circle.[1]

GAUSS too, busied himself for a good many years with "the geometry of the acute angle", and must no doubt be considered almost a partisan of it. It looks however as if in his inmost mind there was a certain feeling that here, none the less, he was out of the right track. He does not publish anything about the problem; it even looks as if he had destroyed all his notes concerning it before his death, and every time he speaks briefly about it in his letters, he anxiously entreats the receiver to consider it a confidential communication. He is afraid of "the roar of the Boeotians", he says. These Boeotians have no doubt inhabited his own heart. With respect to his many Euclidian works there were no Boeotians. Here there was no philosophy and no obscurity. But as soon as the philosophy of science comes in, he grows not a little vague — as later on RIEMANN and HELMHOLTZ —: He produces all these exact works, and at the

[1] OSTWALD's Klassiker der exakten Wissenschaften Nr. 130: Pangeometrie von N. I. LOBATSCHEFSKY, p. 6.

same time maintains that geometry is an empirical science. He calls it a science of space and says at the same time: About space we know hardly anything. He praises the great importance of intuition in geometry and yet believes that we may act in opposition to its indisputable demands.[1]

A peculiar position is occupied by Taurinus. Amid the general vagueness he is almost quite clear. He deduces one proposition after the other in the domain of the third hypothesis, but he knows perfectly well that he is outside the plane, that so far he is not at all in contradiction to Euclid, and that it is not at all a new kind of geometry he is producing, but only a new chapter of Euclid. He remarks that if we are to have, besides the system of the straight line, also a system for every kind of curved line, then it will be arbitrariness not to adopt an infinity of systems, nor has he, evidently, shared the usual belief that geometry is a real science of space, for he points out that Lambert's natural unit of length after all can only be relative, as we may imagine many different radii of curvation.[1]

Most of the partisans of the new system use as a principal argument for its justification that it contains no contradictions. To this we must however remark that such an argument has only value when taken absolutely. It is in other words quite possible to form consistent errors, about the starting point of which we may say with Lambert: Et hoc si dederis, danda sunt omnia.

It is quite evident that a spheric as well as a pseudospheric geometry devoid of contradictions may be posited. But if we understand by Euclidian mathematics all

[1] About Gauss and Taurinus see the corresponding sections in Engel and Stäckel.

mathematics educible in accordance with his Elements as a base, then two such geometries will come well within Euclidian scope. We should then have before us not two new geometries in the sense of two new systems, but at most two new chapters of Euclidian mathematics.

The New-Mathematicians, however, interpret the case quite differently, and thus the contradictions arise. They do not admit that they have been transferred to the sphere or the pseudo-sphere and so remained within the Euclidian domain, but they believe that they have remained in the plane and thereby discovered something new beyond the Euclidian domain. But this is contradictory. It is a contradiction to call the actually curved lines, which they are forced to construct, straight ones. It is a contradiction to reckon these different kinds of lines as one mathematical concept, they must be kept separate and cannot straightway transform themselves one into the other, as the New-Mathematicians make them. It is a contradiction to keep the term "a plane" when the lines have ceased to be straight, and it would also be a contradiction to accept two kinds of planes, corresponding to the two kinds of "straight lines", as one mathematical concept. It is a contradiction to view the figures, which with Saccheri and Lambert we are forced to construct on examining the two impossible hypotheses, without perceiving that it is simply the perspective representation of a quadrangle on the sphere or on the pseudo-sphere, we have before us. Perhaps the New-Mathematicians might say that they have been anxious to restrict themselves to concepts and not allow intuition to interfere. But it is again a contradiction to believe in concepts without intuitions. It has here been overlooked that intuition can be certain as well as uncertain, and that it is only the uncertain steps of intuition that we have to beware of, while the certain ones cannot be dispensed with and can just as little be denied without disaster. The New-Mathematicians here remind us of the wife in Holberg who being surprised by her husband in a precarious situation tries to defend herself against his reproaches by saying: But my dear! Will you really rather believe your eyes than your own little wife?

To explain away these contradictions will not be easy, but without such a step it seems quite absurd that the clear and natural course has not been chosen: to admit that the audacious supposition of the obtuse or the acute angle has simply transferred us to where these angles are plainly at home.

The two thousand year old so-called contest about the parallel-postulate was after all really neither a fight about the postulate nor for or against Euclid, but rather a dispute about the nature of the straight line. For everybody — Euclid not excepted — must at once admit that for curved lines the postulate does not hold. And in the same way everybody who reasonably relies on his own eyes must with Euclid be convinced that for actually straight lines the postulate must hold good. The problem is therefore: The straight line.

The contradiction came in when it was gradually assumed that the exact, the perfect mathematics dealt directly with the objects of the real world, for concerning these it is, as said, impossible to build perfect science. But with this assumption a vague ambiguity entered into the concept of the straight line, as well as into the mathematical objects on the whole. For whoever believes that there are exact ellipses or conchoids, cylinders or spheres in the real world? But it is always the exact curves, surfaces or bodies that are dealt with in mathematics.

When the romantic period came, which on the whole valued mystery more than clearness, mysticism also to a certain degree got the upper hand of clearness in mathematics, and the mystical interpretation of the results of SACCHERI and LAMBERT was preferred to the plain and natural one. New-Mathematics has on the whole several points of resemblance with the romantic speculative philosophical systems, and might well be termed *the romantic mathematics* in contrast to *the classical*. It is however important to note that on the whole it is the interpretation which is new. The mathematical steps themselves are, apart from mistakes and such like, on the whole Euclidian. Non-Euclidian Mathematics is strictly speaking simply an impossibility, because our Arithmetic, our Algebra, our Logic, our Intuition, in short all the factors concerned, are Euclidian. But it would be very desirable if also the Non-Euclidian *interpretation* would disappear. For it creates vagueness, and clearness is preferable.

And mathematics can afford clearness.

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